JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 19, No. 2, April–June 2005

Adaptive-Weighting Input-Estimation Approach to Nonlinear Inverse Heat-Conduction Problems

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The inverse heat-conduction problem involves surface-heat-flux or heat-source estimation that requires only the temperatures measured at an insulated wall. This problem becomes nonlinear if the thermal properties are temperature-dependent. Innovative adaptive-weighting input-estimation inverse methodology for estimating a time-varying unknown heat source from a nonlinear thermal system is presented. This algorithm includes the extended Kalman filter (EKF) and the recursive least-square estimator. EKF recursively estimates the interior temperature of a body under a system involving noisy measurement and modeling errors. During the EKF estimation procedure, an important regression equation between the observable bias residual innovation and the thermal unknown is provided. Based on this regression model, a recursive least-square estimator weighted by the adaptive weighting factor is proposed to estimate these unknowns, defined as the input. The Kalman tuning parameter is used first to analyze the interactive relationship between measurement noise and modeling-error variance. The superior capabilities of the proposed algorithm are demonstrated through two simulated examples with different types of time-varying heat sources as the unknown inputs.

Nomenclature

a	=	coefficient
$\tilde{a}$	=	coefficient
B	=	sensitivity matrix
b	=	coefficient
$\tilde{b}$	=	coefficient
C_p	=	heat capacity
D	=	total number of time steps
F_T	=	Jacobian matrix
F_φ	=	Jacobian matrix
G	=	coefficient matrix
K	=	Kalman gain
k	=	time (discretized)
L	=	strip length
n	=	total number of spatial nodes
P	=	filter error covariance matrix
P_b	=	error covariance
Q	=	process noise covariance
Q_d	=	process noise covariance
R	=	measurement error covariance
s	=	innovation covariance
T	=	temperature
t	=	time
t_f	=	final time
t_i	=	discrete time
x	=	spatial coordinate
z	=	observation state
Γ	=	input matrix
γ	=	forgetting factor
δ	=	Dirac delta function
κ	=	thermal conductivity
ν	=	residual of innovation sequence
ρ	=	density
Φ	=	state transition matrix
ψ	=	coefficient matrix
φ	=	heat source
σ	=	standard deviation

Superscripts

T	=	transpose of matrix
$\bar{}$	=	estimated by filter
$\hat{}$	=	estimated
$*$	=	nominal denote

Introduction

INVERSE problems in heat conduction have been of interest to many researchers in recent years. It is sometimes necessary to calculate transient surface heat flux and surface temperature from a temperature measured at some location inside a body. Typical examples include skin-surface heat-flux estimation for a reentry vehicle, evaluating and testing new rocket heat-shield materials. If the thermal properties are functions of temperature, as considered in this paper, the inverse problem becomes nonlinear. One of the first papers on this subject was written by Stolz.[1] The inverse approaches includes constructing a computational model of the energy equation, evaluating the temperatures at which the sensors are located, and selecting the optimum algorithm to minimize an objective function characterized by the sum of square errors between the measured and corresponding calculated values. Owing to the diffuse nature of the heat-conduction problem and the noisy measurements involved, his procedure was unstable if the time intervals were made too small. Therefore, much smaller time steps were used by Beck by utilizing the least-squares method.[2] Regularization techniques were introduced to effectively suppress instability and sensitivity in the estimated results, although providing the desired accuracy. The extensively used regularization methods[3–5] notably are all batch forms. In this present work an on-line, rather than batch form was used. The algorithm includes extended Kalman filtering (EKF) and the recursive least-square estimator (RLSE). The Kalman filter works together with the recursive least-square estimator to recursively estimate thermal unknowns. This approach has been successfully employed to solve inverse heat conduction problems (IHCPs) in recent

Presented at the AIAA Proceedings of the 2000 National Heat Transfer Conference, Pittsburgh, PA, 20–22 August 2000; received 2 March 2004; revision received 27 May 2004; accepted for publication 1 June 2004. Copyright © 2004 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0887-8722/05 $10.00 in correspondence with the CCC.

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years.[6–8] This application can be extremely useful in actual thermal systems that need to immediately identify thermal unknowns such as a heat source or heat flux or even thermal system properties. For instance, a high-temperature flame jet impacts on an active body surface and the magnitude of this heat flux must be estimated on line for design, control, or safety considerations. This input-estimation approach has been proven simple in concept and less computationally complex in implementation. The most significant feature is that the unknown is a real-time estimate followed by the measurement, taken at the current time. This on-line input-estimation algorithm does not require accumulating measurements after they are processed. Instead, we have a running estimate so that the measurements can be discontinued after the estimate no longer changes an appreciable amount from data point to data point. The linear input-estimation approach consists of two recursive estimators and the Kalman filter for estimating temperatures from noisy measurement. Here the heat-conduction state-space representation is used as the basic model. The Kalman filter as a predictor–corrector algorithm and the state as the temperature predictor will be discussed. Because there are thermal unknowns in the system, we are not able to compute the state unless the unknowns are available. However, in using the Kalman-filter technique, innovations were introduced to derive a linear regression relationship between the observable biased residual–innovation sequence and the unknowns. This observable residual innovation is defined as the difference between the temperature measurement and bias-free temperature estimated from the Kalman filter. Based on this regression model, the RLSE is presented to estimate these unknowns. From previous researches, we found that many inverse methods are limited to the linear heat-transfer system. Few are applicable to nonlinear problems.[9–14] The input estimation algorithm for solving nonlinear inverse heat-conduction problems was written by Tuan.[15] The reason is that many heat-transfer processes in practice are nonlinear rather than linear. Coupling the nonlinear ties with noisy measurement data makes the inverse problem challenging. In this paper, the extended Kalman-filter algorithm is developed for a discrete nonlinear heat conduction problem. The system equation is linearized using the new estimate as soon as it becomes available. The RLSE proposed in previous research is then applied.[8] The RLSE weighted by the adaptive-weighting factor is proposed to estimate unknowns that are defined as the input. The Kalman tuning parameter (Q/R) is used to analyze the interactive relationship between the measurement and modeling error variance. Through the simulated validation, the proposed algorithm is used to innovatively solve the nonlinear IHCP.

Problem Formulation

Assume that there is a homogeneous thermal conductor with an unknown heat source $\varphi(t)$ applied at $x = x_b$. For convenience assume that the boundary at $x = 0$, $x = L$ is insulated, The measured temperature $z(t)$ is from a thermocouple at $x = L$. Figure 1 shows the geometry and coordinates. The mathematical formulation of the one-dimensional, transient, nonlinear heat-conduction problem can be generalized as

$$\frac{\partial}{\partial x}\left(\kappa(T)\frac{\partial T}{\partial x}\right) + \varphi(t)\delta(x - x_b) = \rho C_p(T)\frac{\partial T}{\partial t}$$

$$0 < x < L, \qquad 0 < t \le t_f \tag{1a}$$

$$-\kappa(T)\frac{\partial T}{\partial x} = 0 \qquad x = 0 \tag{1b}$$

$$-\kappa(T)\frac{\partial T}{\partial x} = 0 \qquad x = L \tag{1c}$$

$$T(x, 0) = f(x) \qquad t = 0 \tag{1d}$$

$$z(t) = T(x, t) + \nu(t) \quad x = L \quad \text{(noise-corrupted measurements)} \tag{2}$$

where T is the temperature distribution as a function of x and t, δ is the Kronecker delta function, κ(T) is the thermal conductivity, and $\rho C_p(T)$ is the volumetric heat capacity, which will change with the variation in temperature. $\varphi(t)$ is the unknown heat source at the position $x = x_b.\nu(t)$ is white Gaussian noise with a zero mean value. For simplification, the following assumptions are made:

$$\kappa(T) = a + bT, \qquad \rho C_p(T) = C(T) = c = \text{constant} \tag{3}$$

$$\kappa(T)/c = \tilde{a} + \tilde{b}T \tag{4}$$

Substitute Eqs. (3) and (4) into Eq. (1a) to obtain

$$\frac{\partial T}{\partial t} = (\tilde{a} + \tilde{b}T)\frac{\partial^2 T}{\partial x^2} + \tilde{b}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{c}\varphi(t)\delta(x - x_b) \tag{5}$$

Based on the finite-difference method, the relation between the temperature $T(x, t)$ and heat source $\varphi(t)$ can be developed further. In this study, the space derivative was applied using the central-difference approximation. This is according to the method described by D'Souza.[16] Equation (5) then becomes

$$\dot{T}_i(t) = (\tilde{a} + \tilde{b}T_i)\left(\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2}\right) + \tilde{b}\left(\frac{T_{i+1} - T_{i-1}}{2\Delta x}\right)^2$$

$$+ \frac{1}{c}\varphi(t)\delta(x_i - x_b) = \frac{\tilde{a}}{(\Delta x)^2}(T_{i+1} - 2T_i + T_{i-1})$$

$$+ \frac{\tilde{b}}{(\Delta x)^2}\left(T_{i+1}\cdot T_i - 2T_i^2 + T_i\cdot T_{i-1}\right) + \tilde{b}\left(\frac{T_{i+1} - T_{i-1}}{2\Delta x}\right)^2$$

$$+ \frac{1}{c}\varphi(t)\delta(x_i - x_b), \qquad \text{for} \qquad i = 0, 1, \ldots, N \tag{6}$$

Substitute the boundary condition into Eq. (6) and account for the process noise input.[15] The nonlinear continuous time state equation can be written as

$$\dot{T}(t) = f[T(t), \varphi(t), t] + G(t)\omega(t) \tag{7}$$

$$z(t) = h[T(t), \varphi(t), t] + \nu(t) \qquad t = t_i, \qquad i = 1, 2, \ldots$$

$$T(t) = \{T_0(t), T_1(t), T_2(t), \ldots, T_N(t)\}^T \tag{8}$$

We shall assume that measurements are only available at specific values of time, at $t = t_i$, $i = 1, 2, \ldots$; thus, our measurement equation will be treated as a discrete-time equation, whereas our state equation will be treated as a continuous-time equation. State vector $T(t)$ is temperature $n \times 1$, $n = N + 1$; the heat source $\varphi(t)$ is an unknown input; $z(t)$ is measurement temperature; $\dot{T}(t)$ is short for $\mathrm{d}T(t)/\mathrm{d}t$; nonlinear functions f and h may depend both implicitly and explicitly on t; $\omega(t)$ is a continuous-time white-noise process; $\nu(t_i)$ is a discrete-time white-noise sequence; and $\omega(t)$ and $\nu(t_i)$ are mutually uncorrelated at all $t = t_i$.

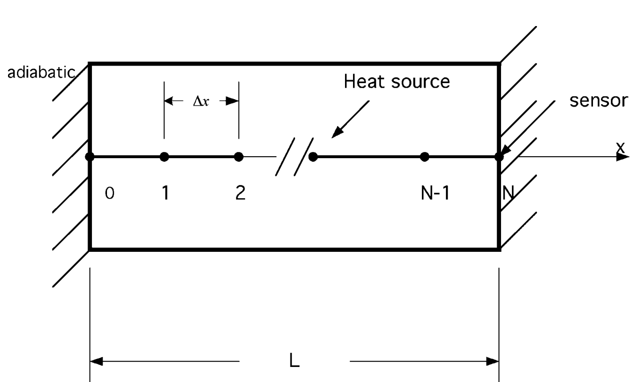

Fig. 1 Geometry and coordinates.

A nonlinear dynamical system is linearized about the nominal values of its state vector and control input. Given a nominal input $\varphi^*(t)$ and assuming that a nominal trajectory $T^*(t)$ and its associated nominal measurement satisfy the following nominal system model:

$$T(t) = f[T^*(t), \varphi^*(t), t] \tag{9}$$

$$z^*(t) = h\lfloor T^*(t), \varphi^*(t), t\rfloor \qquad t = t_i, \qquad i = 1, 2, \ldots \tag{10}$$

If $T^*(t)$ exists and

$$\delta T(t) = T(t) - T^*(t), \qquad \delta\varphi(t) = \varphi(t) - \varphi^*(t)$$

$$\delta z(t) = z(t) - z^*(t) \tag{11}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta T(t) = \delta\dot{T}(t) = \dot{T}(t) - \dot{T}^*(t) = f[T(t), \varphi(t), t]$$

$$+ G(t)\omega(t) - f[T^*(t), \varphi^*(t), t] \tag{12}$$

When $f[T(t), \varphi(t), t]$ is expanded in a Taylor series about $T^*(t)$ and $\varphi^*(t)$, we obtain

$$f[T(t), \varphi(t), t] = f\lfloor T^*(t), \varphi^*(t), t\rfloor + F_T\lfloor T^*(t), \varphi^*(t)t\rfloor\delta T(t)$$

$$+ F_\varphi\lfloor T^*(t), \varphi^*(t), t\rfloor\delta\varphi(t) + \text{higher-order terms} \tag{13}$$

where F_T and F_φ are $n \times n$ and $n \times 1$ Jacobian matrices,[17] that is,

$$F_T[T^*(t), \varphi^*(t), t] = \begin{bmatrix} \dfrac{\partial f_1}{\partial T_1^*} & \cdots & \dfrac{\partial f_1}{\partial T_n^*} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial T_1^*} & \cdots & \dfrac{\partial f_n}{\partial T_n^*} \end{bmatrix} \tag{14}$$

$$F_\varphi[T^*(t), \varphi^*(t), t] = \begin{bmatrix} \dfrac{\partial f_1}{\partial \varphi_1^*} \\ \vdots \\ \dfrac{\partial f_n}{\partial \varphi_1^*} \end{bmatrix} \tag{15}$$

Substituting Eq. (13) into Eq. (12) and neglecting the higher-order terms, we obtain the following perturbation state equation:

$$\delta\dot{T}(t) = F_T[T^*(t), \varphi^*(t), t]\delta T(t)$$

$$+ F_\varphi[T^*(t), \varphi^*(t), t]\delta\varphi(t) + G(t)\omega(t) \tag{16}$$

The approximation is considered good if the difference between the nominal and actual solutions can be described using a linear differential equation system called the linear perturbation state equation. When $h[T(t), \varphi(t), t]$ is expanded in a Taylor series about $T^*(t)$ and $\varphi^*(t)$, we obtain

$$h[T(t), \varphi(t), t] = h[T^*(t), \varphi^*(t), t] + H_T[T^*(t), \varphi^*(t), t]\delta T(t)$$

$$+ H_\varphi[T^*(t), \varphi^*(t), t]\delta\varphi(t) + \text{higher-order terms} \tag{17}$$

$$H_T[T^*(t), \varphi^*(t), t] = \begin{bmatrix} \dfrac{\partial h_1}{\partial T_1^*} & \cdots & \dfrac{\partial h_1}{\partial T_n^*} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial h_n}{\partial T_1^*} & \cdots & \dfrac{\partial h_n}{\partial T_n^*} \end{bmatrix} \tag{18}$$

$$H_\varphi[T^*(t), \varphi^*(t), t] = \begin{bmatrix} \dfrac{\partial h_1}{\partial \varphi_1^*} \\ \vdots \\ \dfrac{\partial h_n}{\partial \varphi_1^*} \end{bmatrix} \tag{19}$$

where H_T are H_φ are $n \times n$ and $n \times 1$ Jacobian matrices. Substituting Eq. (17) into Eq. (11) and neglecting the higher-order terms, we obtain the following perturbation measurement equation:

$$\delta z(t) = H_T(T^*(t), \varphi^*(t), t)\delta T(t) + H_\varphi(T^*(t),$$

$$\varphi^*(t), t)\delta\varphi(t) + v(t) \qquad t = t_i, \qquad i = 1, 2, \ldots \tag{20}$$

If $\varphi(t)$ does not depend on $T(t)$ then usually $\varphi(t) = \varphi^*(t)$, in which case $\delta\varphi(t) = 0$; we also have the following discretized perturbation state-variable model associated with a linearized version of the original nonlinear state-variable model:

$$\delta T(k+1) = \Phi(k+1, k; ^*)\delta T(k)$$

$$+ \psi(k+1, k; ^*)\delta\varphi(k) + \omega_d(k) \tag{21}$$

$$\delta z(k+1) = H_T(k+1; ^*)\delta T(k+1)$$

$$+ H_\varphi(k+1; ^*)\delta\varphi(k+1) + v(k+1) \tag{22}$$

where

$$\Phi(k+1, k; ^*) = I + F_T\lfloor T^*(k), k\rfloor\Delta t \tag{23}$$

$$\psi(k+1, k; ^*) = F_\varphi\lfloor\varphi^*(k), k\rfloor\Delta t \tag{24}$$

$\omega_d(k)$ is a discrete-time white Gaussian sequence

$$\omega_d(k) = \int_{t_k}^{t_{k+1}} \Phi\Big(t_{k+1}, \tau; ^*\Big)G(\tau)\omega(\tau)\,\mathrm{d}t$$

and the process noise covariance Q is

$$E\{\omega_d(k)\omega_d'(k)\} \cong Q_d(k+1, k; ^*)$$

$$= \int_{t_k}^{t_{k+1}} \Phi\Big(t_{k+1}, \tau; ^*\Big)G(\tau)Q(\tau)G'(\tau)\Phi'\Big(t_{k+1}, \tau; ^*\Big)\,\mathrm{d}\tau$$

$$= G_k Q_k G_k'\Delta t = Q_d(k) = Q \tag{25}$$

where the Δt is the sampling time step used to obtain the discretized perturbation state-variable model, Eqs. (21) and (22). An extended Kalman filter combined with a recursive least-square method is used to extract the unknowns in the heat source.

Recursive Input-Estimation Approach

The on-line inverse methodology based on the Kalman filtering technique and a weighted recursive least-square estimator with an adaptive weighting factor accurately estimates the unknown heat flux or heat source. The purpose of the Kalman filter is to generate the recursive innovation sequence. Because this sequence contains bias or systematic errors caused by implicit unknown time-variant inputs, and variance or random errors caused by measurement errors, the real-time least-square method can be used to identify any system error induced by the unknown time-variant input. The recursive input-estimator methodology uses the extended Kalman filter instead of a conventional Kalman filter.

Suppose that $T^*(t)$ is given a priori; then we can compute the predicted, filtered, or smoothed estimates for $\delta T(k)$ by applying all of our previously derived estimators to the discretized perturbation state-variable model in Eqs. (21) and (22). We can precompute $T^*(t)$ using the Kalman filter to solve the nominal differential Eq. (9). The relinearized KF is based only on the discretized perturbation state-variable model. Relinearized KF divergence often occurs, but it does not use the nonlinear nature of the original system in an active manner. The extended Kalman filter relinearizes the nonlinear system about each new estimate as it becomes available. The EKF was developed in predictor–corrector format.[18] Its prediction equation was obtained by integrating the nominal differential equation for $T^*(t)$, from t_k to t_{k+1}. To do this, we needed to know how to choose $T^*(t)$ for the entire interval of time $t_1 \in [t_k, t_{k+1}]$. Let t_1 be an arbitrary value of t lying in the interval between t_k and t_{k+1};

when $\delta\hat{T}(t/t_k)=0$, for all $t\in[t_k, t_{k+1}]$, then $T^*(t)=\hat{T}(t/t_k)$. We can precompute Eq. (21) using the Kalman filter to solve

$$\delta\hat{T}(t_l/t_k) = \Phi\big(t_l, t_k;^*\big)\delta\hat{T}(t_k/t_k) = \Phi\big(t_l, t_k;^*\big)[\hat{T}(k/k) - T^*(k)] \quad (26)$$

In the EKF we set $T^*(k)=\hat{T}(k/k)$; thus, when this is done,

$$\delta\hat{T}(t_l/t_k) = 0 \quad (27)$$

Because $T^*(t)$ is the solution for Eq. (9), for all times $t\in[t_k, t_{k+1}]$, $T^*(t)=\hat{T}(t/t_k)$; thus, in Eq. (9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{T}(t/t_k) = f\big[\hat{T}(t/t_k), \varphi^*(t), t\big] \quad (28)$$

Integrating this equation from $t=t_k$ to $t=t_{k+1}$, we obtain

$$\hat{T}(k+1/k) = \hat{T}(k/k) + \int_{t_k}^{t_{k+1}} f\big[\hat{T}(t/t_k), \varphi^*(t), t\big]\,\mathrm{d}t \quad (29)$$

Equation (29) is the EKF prediction equation. Observe that the nonlinear nature of the system's state equation is used to determine $\hat{T}(k+1/k)$. The integral in Eq. (29) is evaluated using numerical integration formulas that are initialized by $f[\hat{T}(t_k/t_k), \varphi^*(t_k), t_k]$.

The corrector equation for $\hat{T}(k+1/k+1)$ is obtained from the Kalman filter associated with the discretized perturbation state-variable model in Eqs. (21) and (22) and is

$$\delta\hat{T}(k+1/k+1) = \delta\hat{T}(k+1/k) + K(k+1;^*)\big[\delta z(k+1) - H_T(k+1;^*)\delta\hat{T}(k+1/k) - H_\varphi(k+1;^*)\delta\varphi(k+1)\big] \quad (30)$$

As a consequence of relinearizing about $\hat{T}(k/k)$, we know that

$$\delta\hat{T}(k+1/k) = 0 \quad (31)$$

$$\delta\hat{T}(k+1/k+1) = \hat{T}(k+1/k+1) - T^*(k+1) = \hat{T}(k+1/k+1) - \hat{T}(k+1/k) \quad (32)$$

$$\delta z(k+1) = z(k+1) - z^*(k+1) = z(k+1) - h[T^*(k+1), \varphi^*(k+1), k+1] = z(k+1) - h[\hat{T}(k+1/k), \varphi^*(k+1), k+1] \quad (33)$$

Substituting these three equations into Eq. (30), we obtain

$$\hat{T}(k+1/k+1) = \hat{T}(k+1/k) + K(k+1;^*) \times \big\{z(k+1) - h[\hat{T}(k+1/k), \varphi^*(k+1), k+1] - H_\varphi(k+1;^*)\delta\varphi(k+1)\big\} \quad (34)$$

which is the EKF correction equation. To compute $\hat{T}(k+1/k+1)$, we must compute the gain matrix $K(k+1;^*)$. This matrix, and its associated $P(k+1/k;^*)$ and $P(k+1/k+1;^*)$ matrices, depend on the nominal $T^*(t)$ that results from the prediction, $\hat{T}(k+1/k)$. Observe, from $T^*(t)=\hat{T}(t/t_k)$, that $T^*(k+1)=\hat{T}(k+1/k)$ and that the argument of K in the correction equation is $k+1$; these three quantities are computed from

$$K(k+1;^*) = P(k+1/k;^*)H_T^T(k+1;^*)\big[H_T(k+1;^*) \times P(k+1/k;^*)H_T^T(k+1;^*) + R(k+1)\big]^{-1} \quad (35)$$

$$P(k+1/k;^*) = \Phi(k+1,k;^*)P(k/k;^*)\Phi^T(k+1,k;^*) + Q_d(k+1,k;^*) \quad (36)$$

$$P(k+1/k+1;^*) = [I - K(k+1;^*)H_T(k+1;^*)] \times P(k+1/k;^*) \quad (37)$$

Remember that in these three equations $*$ denotes the use of $T^*(k+1)=\hat{T}(k+1/k)$. Equations (29), (34), and (35–37) are called the EKF equations. $T(k)$ is the state vector, $\Phi(k+1,k;^*)$ is the state transition matrix, $\varphi(k)$ is the heat source, $\omega_d(k)$ is the Gaussian process noise vector with assumptions of zero mean value and variance $E\{\omega_d(k)\omega_d^T(j)\}=Q_d\delta_{kj}$, and δ_{kj} is the Kronecker delta function. $z(k)$ is the observed state at time k; $H_T=[0\ \ 0\ \ \cdots\ \ 1]$ is the measured matrix; $\nu(k)$ is the measured noise with assumptions of zero mean value and variance $E\{\upsilon(k)_\upsilon^T(j)\}=R\delta_{kj}$. Define

$$s(k+1;^*) = H_T(k+1;^*)P(k+1/k;^*)H_T^T(k+1;^*) + R(k+1) \quad (38)$$

Because we are only interested in estimating the magnitude of the heat source, according to the Tuan study,[6] the recursive least-squares estimates (RLSEs) are

$$B(k+1) = H[\Phi M(k) + I]\psi \quad (39)$$

$$M(k+1) = [I - K(k+1)H][\Phi M(k) + I] \quad (40)$$

$$K_b(k+1) = \gamma^{-1}P_b(k)B^T(k+1) \times \big[B(k+1)\gamma^{-1}P_b(k)B^T(K+1) + s(k+1)\big]^{-1} \quad (41)$$

$$P_b(k+1) = [I - K_b(k+1)B(k+1)]\gamma^{-1}P_b(k) \quad (42)$$

$$\hat{\varphi}(k+1) = \hat{\varphi}(k) + K_b(k+1)[\bar{z}(k+1) - B(k+1)\hat{\varphi}(k)] \quad (43)$$

where $\hat{\varphi}(k+1)$ is the estimated unknown input heat source, $P_b(k+1)$ is the error covariance of the estimated input vector, $B(k+1)$ and $M(k+1)$ are the sensitivity matrices, $K_b(k+1)$ is the Kalman gain, and $s(k+1)$ is the covariance of the residual. The computation flow-chart for $\hat{\varphi}(k+1)$ given in Fig. 2. $\gamma(k)$ is an adaptive weighting factor that may be presented as[8]

$$\gamma(k) = \begin{cases} 1 & |\bar{Z}(k)| \le \sigma \\ \sigma/|\bar{Z}(k)| & |\bar{Z}(k)| > \sigma \end{cases} \quad (44)$$

where σ is the standard deviation of the measurement errors, $\sigma=\sqrt{R}$. Using $\gamma(k)$ to replace the γ in Eqs. (41) and (42), when $\gamma=1$, the above algorithm reduces to the usual sequential least squares, which is suitable only for a constant-parameter system. In this case, the correction gain $K_b(k+1)$ for updating $\hat{\varphi}(k+1)$ in Eq. (43) diminishes as k increases, which allows $\hat{\varphi}(k+1)$ to converge to the true constant value. In the time-varying case, however, we wish to prevent $K_b(k+1)$ from decreasing to zero. This is accomplished by introducing the factor $\gamma(k)$. For $0<\gamma\le 1$, $K_b(k+1)$ is effectively prevented from shrinking to zero. Hence, the corresponding algorithm can continuously preserve its updating ability. However, the inherent data-truncation effect brought about by $\gamma(k)$ causes variance increases in $\hat{\varphi}(k+1)$ in the estimation problem resulting from noise. Thus, it is constructed with adaptive weighting, allowing it to handle different input response statuses easily.

Results and Discussion

To illustrate the accuracy of the algorithm including EKF and RLSE, we consider a plate with length $L=1.0$ m. Two boundary conditions are adiabatic at both ends. The initial temperature is $T_0=0°$C. The following physical quantities were used in the calculation: thermal conductivity $\kappa(T)=2+0.01T$ kw/m°C, $\rho C_p=2.0$ kJ/m^3°C. The total measurement was taken over a period of 100 s with a measurement time step of $\Delta t=0.005$ s. The space step was $\Delta x=0.2$ m. A thermocouple was placed at the surface $x=L=1.0$ m. The unknown heat source $\varphi(t)$ was placed at the center of the plate. We studied test cases using simulated measured temperature; the simulated temperature data were generated

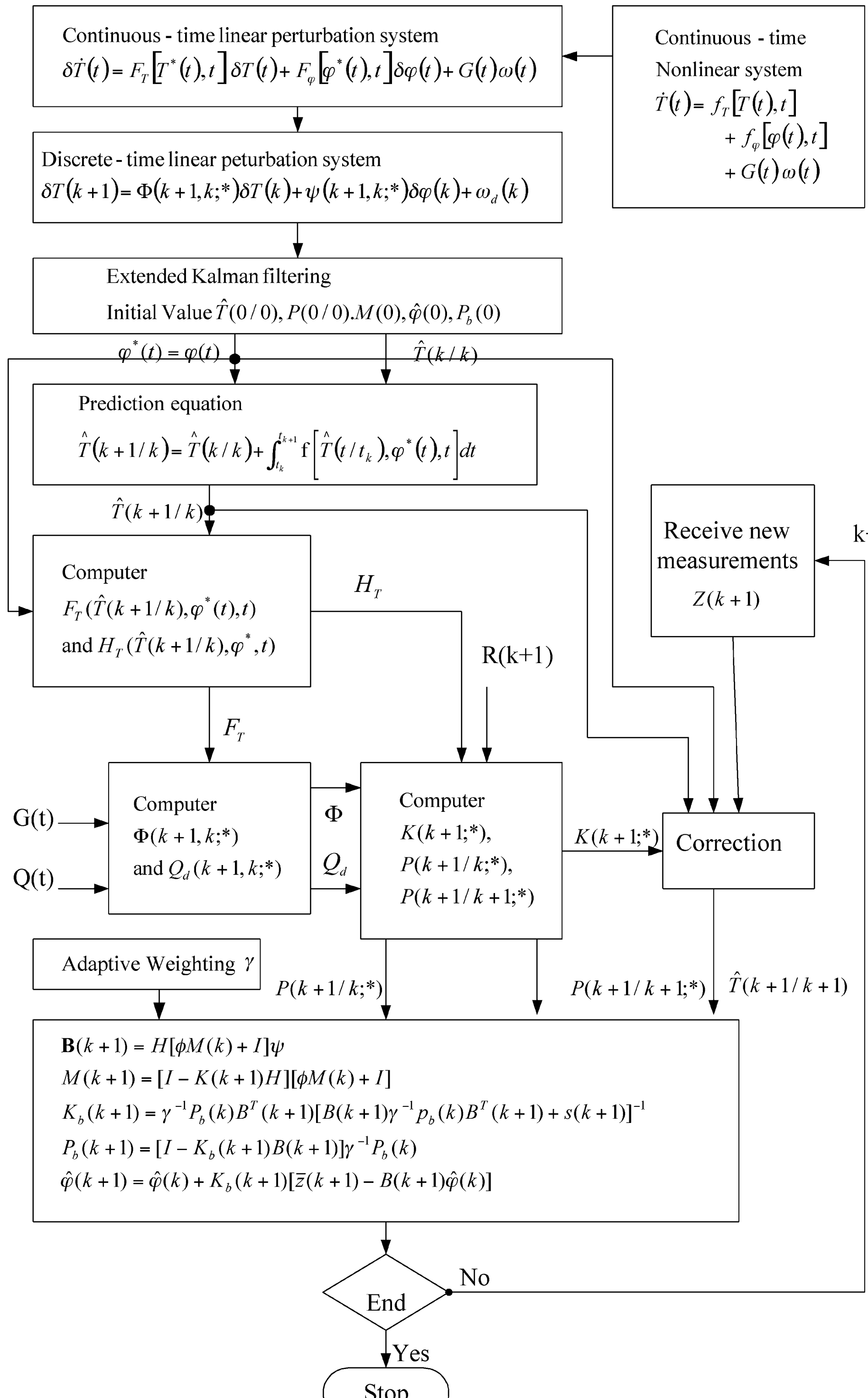

Fig. 2 Flowchart of the adaptive-weighting input-estimation algorithm.

by a Runge–Kutta method, solving the direct problem for a specified functional form for the unknown heat source.[15] The initial conditions for the input estimator were given by $\bar{T}(-1/-1) = [0\ 0\ \cdots\ 0]^T$ and $P(-1/-1) = \mathrm{diag}\lfloor 10^{10} \rfloor$ for the extended Kalman filter. The RLSE algorithm initial conditions were given by $\hat{\varphi}(-1) = 0$, $P_b(-1) = 10^8$, and $M(-1)$ set using a zero matrix.

Because $P(-1/-1)$ and $P_b(-1)$ are normally not known, the estimator was initialized with $P(-1/-1)$ and $P_b(-1)$ as very large numbers, such as 10^{10} and 10^8, respectively. This had the effect of treating the initial errors as very large. The estimator would then ignore the first few initial estimates.[19] The Kalman filter for the recursive input-estimation algorithm requires exact knowledge of the process-noise covariance Q and the measurement-noise covariance R. R depends on the sensor measurements. Both the Q value in the filter and the γ value in the sequential least-squares approach interactively affect the fast adaptive capability for tracking the time-varying parameter. The test heat-source profile is given in the following.

Example 1: Square waveform in $\varphi(t)$ kW/m^3:

$$\varphi(t) = \begin{cases} 0 & 0 \le t < 15,\ 35 < t < 55,\ 75 < t \le t_f \\ 15 & 15 \le t \le 35 \\ 10 & 55 \le t \le 75 \end{cases} \tag{45}$$

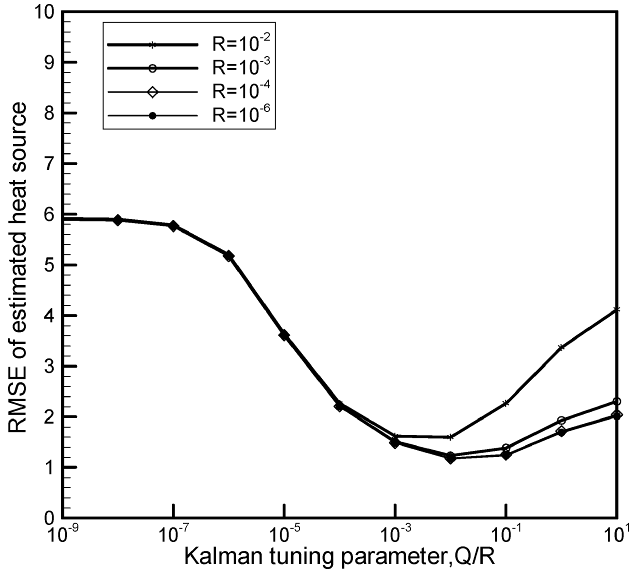

Fig. 3 Root-mean-square error of estimated square wave as input heat source vs Kalman tuning parameter *Q/R* with different *R*.

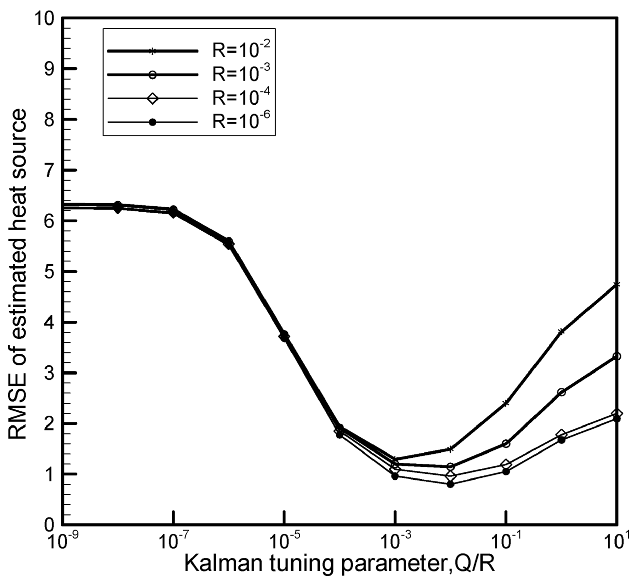

Fig. 4 Root-mean-square error of estimated sine wave mixed with triangle wave as input heat source vs Kalman tuning parameter *Q/R* with different *R*.

Example 2: Combined sine and triangular waveform in $\varphi(t)$ kW/m^3:

$$\varphi(t) = \begin{cases} 20 & 0 \le t < 5, 25 \le t \\ & < 55, 75 < t \le t_f \\ 20 \times \{1 + \sin[0.157(t-5)]\} & 15 \le t \le 35 \\ -2t + 170 & \end{cases} \quad (46)$$

Figures 3 and 4 show the root mean square error of the estimated heat source under different measurement error variances ($R = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-6}). The Kalman tuning parameter Q/R is variable from 10^{-9} to 10. The root-mean-square error (RMSE) for the estimated heat source is defined as[14]

$$\mathrm{RMSE} = \left[\frac{1}{D}\sum_{k=1}^{D}(\varphi_k - \hat{\varphi}_k)^2\right]^{\frac{1}{2}} \quad (47)$$

where D is the total number of time steps in the results from the RMSE, the measurement-error variance's effect on estimate resolution, If R is very large, the RMSE will be large. In practice, the measurement-error variance can be restricted by using an extremely accurate sensor. If the error is very large, the sensor should be replaced. Another important result is that the Kalman tuning parameter Q/R influences the estimate resolution. If Q/R is very small or larger than 1, the estimate accuracy will degraded. If Q/R is chosen between 10^{-3} and 10^{-1}, good estimation will result, the tradeoff between the process-noise covariance Q and the measurement-error variances R can be made by adjusting the Kalman tuning parameter Q/R handling estimation results.

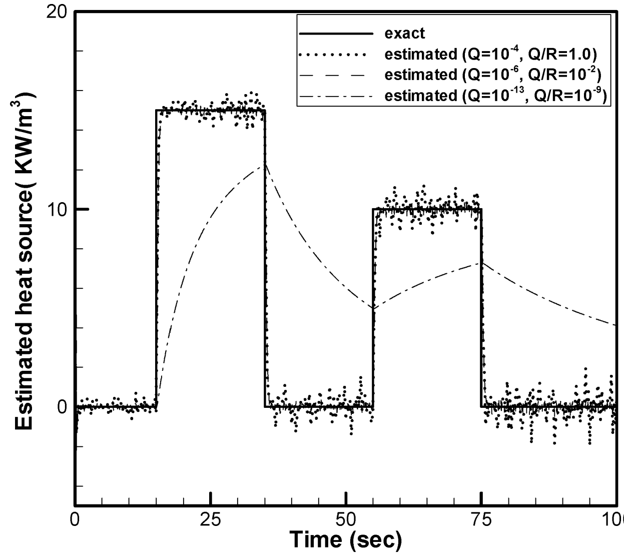

Fig. 5 Effect of various Kalman tuning parameters *Q/R* in the estimation for a square wave as input heat source.

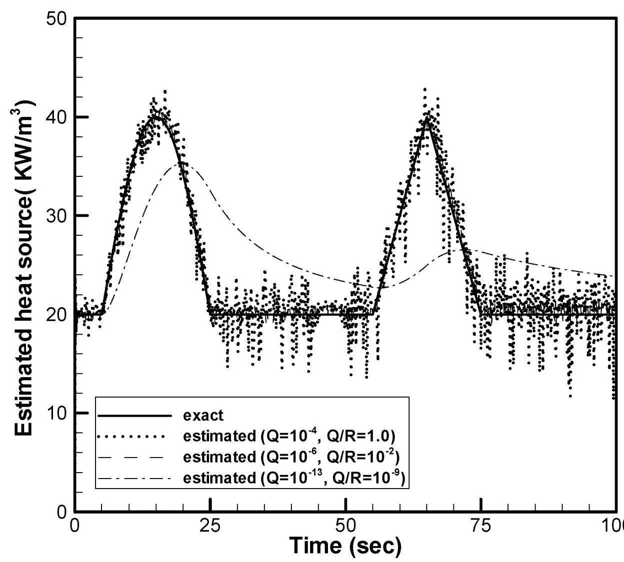

Fig. 6 Effect of various Kalman tuning parameters *Q/R* on the estimation of a sine wave mixed with a triangle wave as input heat source.

Figures 5 and 6 demonstrate the square waveform heat source and combined sine and triangular waveform heat source estimate, respectively. Assume that the measurement error variance $R = 10^{-4}$ is fixed, and the Kalman tuning parameter ($Q/R = 1, 10^{-2}, 10^{-9}$) at the time is variable from 0 to 100 s. When $Q/R = 10^{-2}$, RMSE to be minimum, here we see that as Q increases, Q/R increases, and the filter bandwidth increases. Thus the filter transient performance is faster, but at the cost of more noise in the state estimates. If Q/R is equal to or larger than 1, the fluctuation will become larger. If Q decreases, Q/R decreases, decreasing the bandwidth. The filter transient performance is slower, but noise is filtered. Thus the state

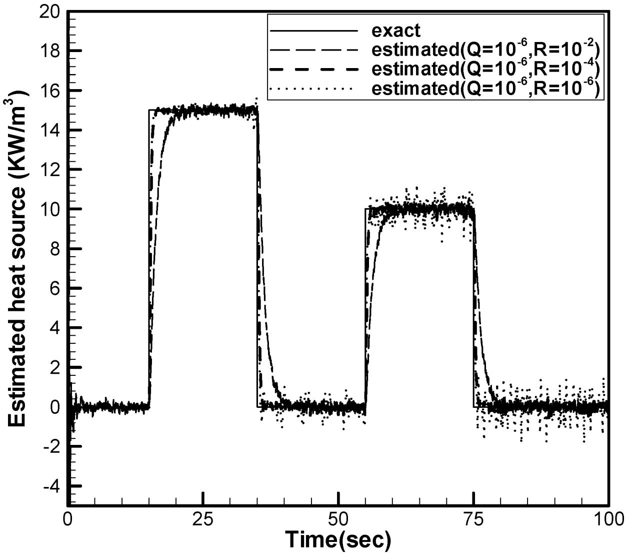

Fig. 7 Estimated square wave as input heat source.

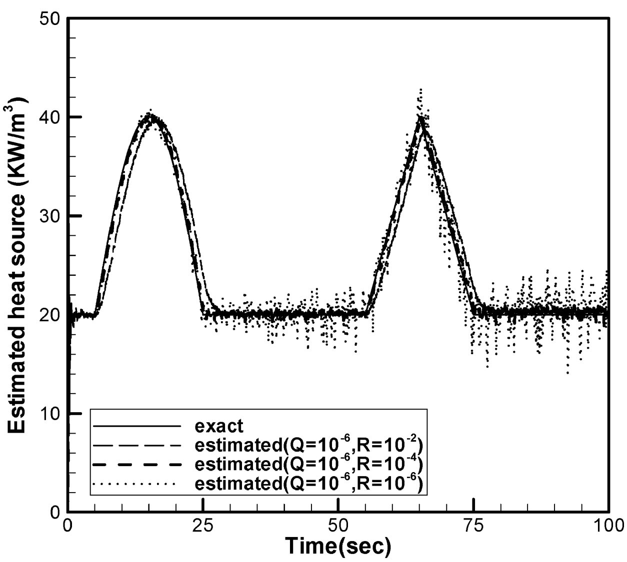

Fig. 8 Estimated sine wave mixed with triangle wave as input heat source.

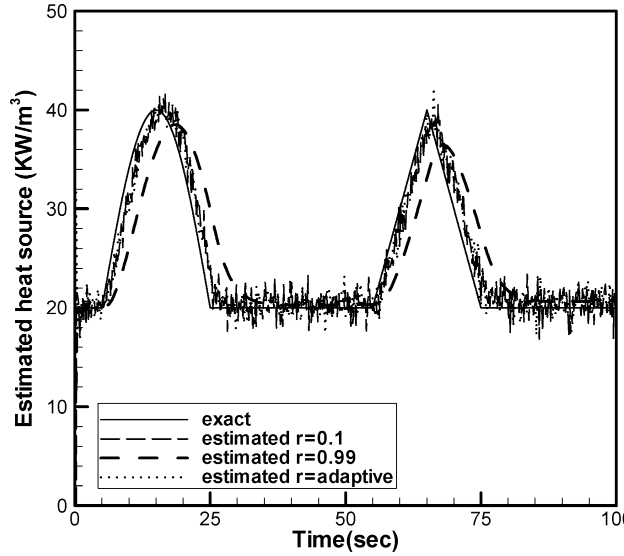

Fig. 9 Comparison of constant-weighting and adaptive-weighting estimated results for example 1 with $R = 10^{-2}$, $Q = 10^{-6}$.

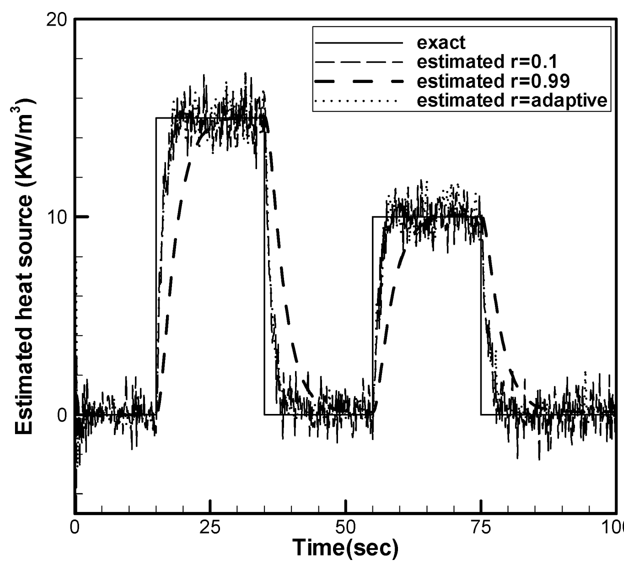

Fig. 10 Comparison of constant-weighting and adaptive-weighting estimated results for example 2 with $R = 10^{-2}$, $Q = 10^{-6}$.

estimates are smoother. However, the proposed algorithm efficiently obtains the exact value of the heat source in the nonlinear inverse heat-conduction problem. Because we linearized the problem, uncertainty exists, and we let Q be the modeling error variance. In Figs. 7 and 8 we demonstrate that when the modeling error variance $Q = 10^{-6}$ is fixed, we obtain different measurement error variances ($R = 10^{-2}$, 10^{-4}, 10^{-6}) at times variable from 0 to 100 s. Here we see that when R is small, the filter transient performance is faster, but at the cost of more noise in the state estimates; the fluctuation will become larger if R increases, decreasing the bandwidth. The filter transient performance is slower, but noise is filtered. The results are similar to those in Ref. 20. When a measurement sensor is chosen, R known, we found that correctly choosing the process noise covariance Q could produce an acceptable estimation result. Because the initial error variance $P(-1/-1)$ is extremely large, the convergence of the algorithm is fast.

Figures 9 and 10 shows comparison-constant-weighting and adaptive-weighting estimated results for Examples 1 and 2 with $R = 10^{-2}$, $Q = 10^{-6}$. simulation results demonstrate the adaptive-weighting input-estimation inverse methodology has good performance in tracking the time-varying unknown heat sources in a nonlinear thermal system.

Conclusions

In this paper, an adaptive-weighting input-estimation inverse methodology was applied to estimating a time-varying unknown heat source in a nonlinear thermal system. This algorithm includes the extended Kalman filter (EKF) and the recursive least-square estimator (RLSE), which recursively estimates the thermal unknowns under a system involving measurement and modeling errors. The capabilities of the proposed algorithm were demonstrated using two simulated examples. The simulated results show that the role of the Kalman tuning parameter influences the estimated results. Even at actual instrument accuracy restrictions or an inaccurate simplified mathematical model, correctly choosing the Kalman tuning parameter Q/R in harmony with the adaptive weighting factor gives that the proposed algorithm has superior estimation ability.

Acknowledgment

The National Science Council of the Republic of China supported this work under Grant NSC-92-2212-E-014-017.

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